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# Boundary Control for Stabilization of Large-Scale Networks through the Continuation Method

Denis Nikitin<sup>1</sup>, Carlos Canudas-de-Wit<sup>1</sup> and Paolo Frasca<sup>1</sup>

**Abstract**—In this work we study a continuation method which transforms spatially distributed ODE systems into PDEs that respect the spatial structure of the original ODE systems. Such PDE description can be used not only for analysis but also for a continuous control design which, being discretized back, results in a nontrivial control law for the original ODE system. In this paper we focus on the continuation for linear systems, including multidimensional inhomogeneous systems and in particular linear networks, showing that such systems can be transformed into general second-order parabolic PDEs. The method is applied to the stabilization of a chain of coupled semiconductor lasers. We obtain a PDE model of this system, design a backstepping-based boundary control to stabilize the obtained PDE and then translate the control policy back to the original laser chain, effectively stabilizing it.

## I. INTRODUCTION

Large-scale networks are often used to describe physical systems, such as urban traffic [1], brain activity [2], power networks [3], robotics formations [4] or epidemic spreading. Entities in these systems have a predefined position in the real-world space, representing individual nodes in the network.

Due to increasing complexity of such systems, methods of simplification of the model are of a great importance. Mostly, the methods of network analysis “forget” information about positions, relying only on interaction topology. Among examples of methods of this type, there is model reduction, in particular clustering of the original network depending on the topology [5], [6] or reduction towards the average state [7], [8].

Some of methods of network analysis represent the original system using continuous models. When every agent in the network is coupled to all others, the method called *population density approach* [9], [10] is used to track the whole probability distribution over all agents’ states in the network. This method is mostly used to model large biological neural networks. More sophisticated topologies can be dealt with a recently emerged theory of graphons which studies graph limits, i.e. structural properties that the graph possesses if the number of nodes tends to infinity while preserving interaction topology, see [11]. Using graphons it is possible to describe any dense graph as a linear operator in continuum space [12]. This method was further used to control large-scale linear networks [13] and to study sensitivity of epidemic networks [14].

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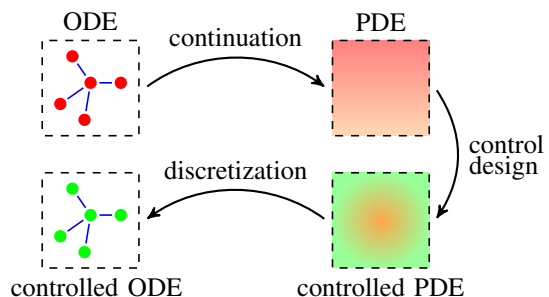


Fig. 1. Proposed framework for control design based on the continuation method and a continuous representation of the system.

Contrary to previous works, we recently developed a method [15] of continuous model transformation which can utilize the intrinsic information of agents’ positions. Our idea is to replace the original spatially distributed ODE system by a continuous PDE whose state and space variables preserve the state and space variables of the original system. We name this method as a *continuation*, since it is exactly opposite to the discretization procedure.

In this paper we focus on the linear network analysis. It is widely known [16], [17] that the Laplacian consensus networks are closely related to the diffusion PDEs. Performing continuation of a general linear spatially-distributed network, we show how the network dynamics can be written as a linear second-order PDE with space-dependent coefficients. In particular we show that several additional properties, such as absence of self-loops, regularity or undirected topology of the network, can simplify the resulting PDE.

The continuation method allows to recover a PDE which describes the same physical system as the original ODE network. The obtained model can be then used for analysis and control. It is possible to design a control for the continuous PDE model which, being discretized back, results in a control law for the original ODE system: the design framework is illustrated in Fig. 1. We demonstrate this continuation framework by designing an explicit control for a chain of coupled semiconductor lasers, suppressing undesirable oscillations.

One could argue: why do we want to use a PDE instead of ODEs, if PDEs are generally considered to be harder to analyze and to control? The answer is that a suitable use of PDEs can lead to explicit and scalable algorithms. Indeed, the centralized computation of feedback control gains for a large-scale linear system with  $N$  agents requires at least  $O(N)$  operations by methods such as ODE-based backstepping

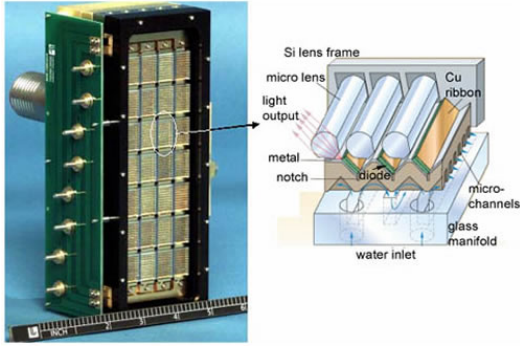


Fig. 2. High-average-power laser-diode 41kW array, composed of 28 silicon monolithic microchannels (SiMMs) each consisting of thousands of diodes. Image from Lawrence Livermore National Laboratory, <https://lasers.llnl.gov/science/photon-science/highpowered-lasers/hapl>, licensed under CC BY-NC-SA 4.0.

[18] and at least  $O(N^3)$  operations by methods like LQR, which require solving a Riccati matrix equation. On the contrary, in the next section we will give an example of such situation where the continuation helps to design a control with gains computed in  $O(1)$  operations. In particular, in case of unstable 1-dimensional PDE we can use the result of [19], where the general second-order linear space-dependent system is stabilized to zero state using backstepping control, based on [20]. The case of two controlled boundaries of 1-dimensional system is covered in [21].

*a) Outline:* The rest of this paper is organized as follows. Section 2 contains our motivating example, the stabilization of a chain of laser diodes: we describe the mathematical model, define the stabilization problem, and solve it by applying the continuation method and the backstepping method. The subsequent sections develop the continuation method. More precisely, Sections 3 and 4 contain a concise but self-contained account of the continuation theory that we developed in [15]: Section 3 regards systems without boundaries and Section 4 regards systems with boundary conditions. Then, Section 5 contains novel continuation results that are tailored to linear networks and that permit the continuation of laser chain model. Section 6 contains some concluding remarks.

## II. SYNCHRONIZATION OF A LASER CHAIN

Our motivating example will be the problem of stabilization of a chain of coupled semiconductor lasers. Coupled laser systems are important for high-precision power transmission applications such as welding, laser surgery or fusion research as well as many others [22]. Recently, large arrays of semiconductor laser diodes were shown to be more power- and cost-efficient compared to single crystal lasers due to lower electrical resistance and optical load [23]. A typical array of coupled lasers is depicted in Fig. 2.

In [24] it was shown that coupling of several Class-B lasers [25] can lead to a resonance effect, greatly increasing intensity comparing to the uncoupled laser system. However, such a system is prone to instabilities: electrical fields of

lasers start to oscillate around the operating point, destroying resonance effect. It was further shown in [24] that these oscillations, up to the first order, are described by coupled Stuart-Landau oscillators.

Stuart-Landau oscillators are prototypical models for Andronov-Hopf bifurcation, and apart from laser applications they are used to describe many oscillatory systems such as electronic oscillators [26] or biological neural networks [27]. Usually in laser analysis Stuart-Landau model describes electrical field of one laser, thus the oscillating behaviour is the desired one. However we base our analysis on [24], where Stuart-Landau model is used to describe deviation from the synchronized steady state: thus, oscillations should be suppressed. Let the deviation of one laser be  $z \in \mathbb{C}$ , then one Stuart-Landau oscillator is described by the evolution equation

$$\dot{z} = (\Gamma + i\Omega - \eta|z|^2)z, \quad (1)$$

where  $\Gamma > 0$  is an excitation gain,  $\Omega$  is a natural frequency and  $\eta \in \mathbb{C}$  is a nonlinear damping coefficient. For  $\Gamma < 0$  the system has one stable equilibrium point  $z = 0$ , while for  $\Gamma > 0$  zero equilibrium point is unstable, and system has a stable limit cycle with frequency  $\Omega$  and with amplitude  $|z| = \sqrt{\Gamma/\eta}$ .

The authors of [22] proposed to design laser hardware having in mind an effect called *amplitude death* to suppress laser electrical field's undesirable oscillations and thus prevent loss of efficiency. This effect appears when many inhomogeneous oscillators are strongly coupled, thus making their limit cycles unstable and the zero fixed point stable. Contrarily to this approach of hardware-designed amplitude death, we propose to use an active feedback stabilization from one boundary to suppress oscillations. We consider here a chain of  $N + 2$  coupled Stuart-Landau oscillators. Let the position of  $i$ -th oscillator for  $i \in \{0, \dots, N + 1\}$  be  $x_i = i\Delta x$  with  $\Delta x = 1/(N + 1)$  being distance between two neighbours, thus  $x_0 = 0$  and  $x_{N+1} = 1$ . The state of  $i$ -th oscillator is  $z_i \in \mathbb{C}$ . We assume that the oscillators on the boundaries are directly controllable, namely the left boundary oscillator has fixed zero state  $z_0 = 0$  and the state of the right boundary oscillator is a control variable  $z_{N+1} := u$ . We also assume that the coupling of lasers is realized by an overlapping of their evanescent fields [28], thus the evolution equation of  $i$ -th oscillator depends on the nearest neighbours' states with gains  $a_{i,i-1}$  and  $a_{i,i+1}$ . Since it is a conservative force,  $a_{i,i+1} = a_{i+1,i}$ , thus the network is undirected. The system is given by

$$\begin{aligned} \dot{z}_i &= (\mu - \eta|z_i|^2)z_i + a_{i,i-1}(z_{i-1} - z_i) + a_{i,i+1}(z_{i+1} - z_i), \\ z_0 &= 0, \quad z_{N+1} = u, \end{aligned} \quad (2)$$

with  $\mu = \Gamma + i\Omega$ . In general, coupling  $a_{i,i+1}$  can be space-dependent. We assume that it is monotone, for example as in case of an increasing electrical permeability of the medium along the laser chain. Therefore, to approximate monotone dependencies we restrict ourselves to a class of coupling gains  $a_{i,i+1} \approx \alpha(x_i - \beta)^2$  with  $\alpha > 0$ ,  $\beta \in \mathbb{R} \setminus [0, 1]$ . Note

that this class includes also homogeneous couplings in case  $\beta \rightarrow \pm\infty$  and  $\alpha \rightarrow 0$  such that  $\alpha\beta^2 \equiv \text{const}$ .

#### A. Continuation process and boundary control design

Since  $\Gamma > 0$ , system (2) has unstable zero equilibrium. Our goal is to design a feedback control law  $z_{N+1} = u(z)$  such that zero solution is stabilized, thus suppressing oscillations. Linearizing system (2) around zero and assuming  $|z_i|$  is small, we get

$$\begin{aligned} \dot{z}_i &= \mu z_i + a_{i,i-1}(z_{i-1} - z_i) + a_{i,i+1}(z_{i+1} - z_i), \\ z_0 &= 0, \quad z_{N+1} = u. \end{aligned} \quad (3)$$

In case of thousands of coupled laser diodes, the implementation of traditional control algorithms for the system (3) would require a lot of computational power. Instead, we can perform continuation to the system (3), the process which will be explained in following sections. This process leads to the following PDE:

$$\frac{\partial z(x,t)}{\partial t} = \mu z(x) + \frac{\partial}{\partial x} \left( \alpha \Delta x^2 (x - \beta)^2 \frac{\partial z(x,t)}{\partial x} \right) \quad (4)$$

for  $x \in (0, 1)$  and with boundary conditions  $z(0, t) = 0$  and  $z(1, t) = u$ .

Although system (4) is formulated in complex domain, one can use backstepping method from [19] to stabilize it. Indeed, the stabilizing controller is given by

$$u := \int_0^1 k(x) z(x, t) dx, \quad (5)$$

and the kernel is found by formula (44) from [19]:

$$\begin{aligned} k(x) &= -\bar{x} \frac{(\mu + c)(1 - \beta)^{3/2}}{\alpha|\beta| (x - \beta)^{5/2}} \times \\ &\times \frac{I_1 \left( \sqrt{(\mu + c)(\bar{y}^2 - \bar{x}^2)/(\alpha\beta^2)} \right)}{\sqrt{(\mu + c)(\bar{y}^2 - \bar{x}^2)/(\alpha\beta^2)}}, \end{aligned} \quad (6)$$

where  $c > 0$  is an adjustable gain,  $I_1(s)$  is the modified Bessel function of order one,  $\bar{x} = -\beta \log(1 - x/\beta)$  and  $\bar{y} = -\beta \log(1 - 1/\beta)$ .

#### B. Control discretization and numerical simulation

Finding a control law for the original ODE system (2) can be easily done by performing a numerical integration of (5) using the trapezoidal rule

$$u := \Delta x \sum_{i=1}^N k(x_i) z_i, \quad (7)$$

Each control gain  $k(x_i)$  can be computed directly in  $O(1)$  operations.

We validated the obtained control law by numerical simulation of system (2) with  $N = 30$  coupled oscillators. We took  $\Gamma = 5$  for excitation gain,  $\Omega = 4$  for natural frequency and  $\nu = 10$  for damping, thus a steady state magnitude of an uncoupled oscillator would be  $|z| = 1/\sqrt{2} \approx 0.7071$ . Further, we took  $\alpha = 5$  and  $\beta = -10$  as parameters for the coupling coefficients  $a_{i,i+1}$ . Due to the coupling, steady

state magnitudes of the network (2) diminish, which can be seen on the graph in Fig. 3(a) depicting simulation of the uncontrolled system (2) with  $u = 0$ . However, the system still oscillates. The oscillations can be suppressed by applying control (7) with kernel (6), where we took  $c = 10$ . Successful suppression is depicted in Fig. 3(b).

### III. CONTINUATION METHOD FOR LINEAR SPACE-INDEPENDENT SYSTEMS

We start presenting the continuation method by considering the simplest class of systems for which the transformation of ODE into PDE can be performed, namely linear ODE systems corresponding to the dynamics of states of nodes, which are aligned on the 1D line in space and depend only on some fixed set of their neighbours. Let the node  $i$  have a state  $\rho_i \in \mathbb{R}$  and a geographical position  $x_i \in \mathbb{R}$  such that for every  $i$  the distance between two consecutive nodes in space is constant,  $x_{i+1} - x_i = \Delta x$  (the assumption of  $\Delta x$  being constant will be relaxed later on). Then the systems of our interest take the form

$$\dot{\rho}_i = \sum_{j=1}^N a_j \rho_{i+s_j}, \quad (8)$$

that is  $\dot{\rho}_i$  linearly depends only on  $N$  neighbouring nodes shifted by  $s_j \in \mathbb{Z}$  for  $j \in \{1..N\}$ , and  $a_j \in \mathbb{R}$  are the system gains, see Fig. 4. This type of systems belongs to the class of linear *spatially invariant systems* [29], which is a natural class for distributed control.

#### A. Discretization

The discretization of PDEs is usually performed by a finite difference method, where the partial derivatives are approximated by finite differences. For example, in the case of Transport ODE,

$$\frac{\partial \rho}{\partial x} \approx \frac{1}{\Delta x} (\rho_{i+1} - \rho_i).$$

This approximation is valid when  $\Delta x$  is small. Indeed, assuming that the solution to PDE is given by a smooth function  $\rho(x)$  and using Taylor series, we can write

$$\rho_{i+1} = \rho(x_{i+1}) = \rho(x_i) + \frac{\partial \rho}{\partial x} \Delta x + \frac{\partial^2 \rho}{\partial x^2} \frac{\Delta x^2}{2} + \dots, \quad (9)$$

where all partial derivatives are calculated in  $x_i$ . Thus, subtracting  $\rho_i$  and dividing by  $\Delta x$ , we get

$$\frac{\partial \rho}{\partial x} = \left[ \frac{1}{\Delta x} (\rho_{i+1} - \rho_i) \right] - \frac{\partial^2 \rho}{\partial x^2} \frac{\Delta x}{2} - \dots, \quad (10)$$

which means that the residual belongs to the class  $O(\Delta x)$  of all functions which go to zero at least as fast as  $\Delta x$ . Thus, taking  $\Delta x$  sufficiently small, one can ensure the arbitrary accuracy of the approximation, provided all the partial derivatives are bounded.

Accuracy can be further increased by taking different points where the function is sampled, called stencil points. For example, writing

$$\rho_{i-1} = \rho(x_{i-1}) = \rho(x_i) - \frac{\partial \rho}{\partial x} \Delta x + \frac{\partial^2 \rho}{\partial x^2} \frac{\Delta x^2}{2} - \dots, \quad (11)$$

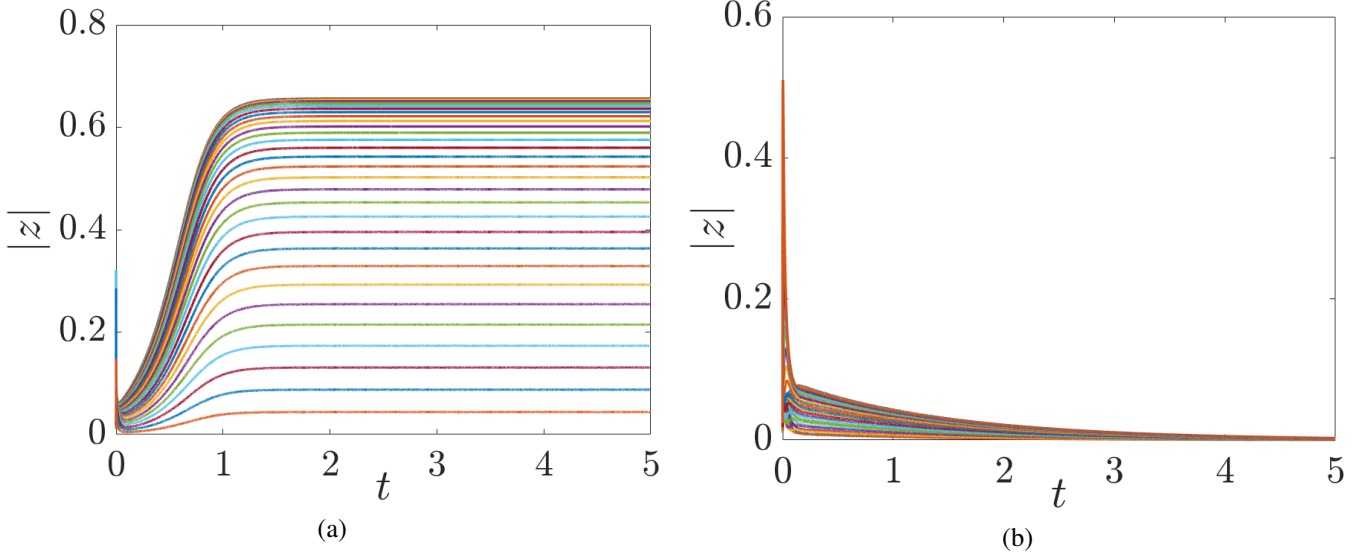


Fig. 3. Numerical simulation of system (2) with  $N = 30$  oscillators with parameters  $\mu = 5 + 4i$ ,  $\eta = 10$ ,  $\alpha = 5$  and  $\beta = -10$ . The absolute values of all states  $|z_i(t)|$  for  $i \in \{1..N\}$  are depicted. Inset (a): uncontrolled system,  $u = 0$ . Inset (b): controlled system with controller (7) with  $c = 10$ .

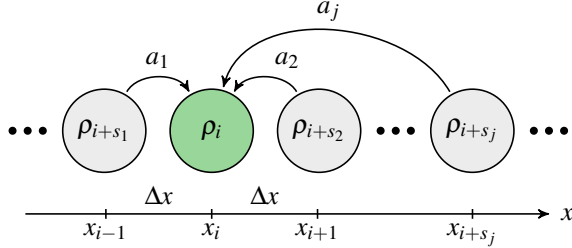


Fig. 4. System of nodes aligned in 1D line with dynamics given by (8) with  $s_1 = -1$  and  $s_2 = 1$ .

subtracting (11) from (9) and dividing by  $2\Delta x$ , we get

$$\frac{\partial \rho}{\partial x} = \left[ \frac{1}{2\Delta x} (\rho_{i+1} - \rho_{i-1}) \right] - \frac{\partial^3 \rho}{\partial x^3} \frac{\Delta x^2}{6} + \dots \quad (12)$$

Thus, using stencil points  $\{i-1, i+1\}$  to approximate the first-order derivative in the point  $i$  the obtained residual belongs to the class  $O(\Delta x^2)$ , which means that this discretization of the Transport PDE is accurate to the second order.

### B. Continuation

Essentially the same process can be applied to the equation (8) to get the PDE version. For every term in the summation in (8) we can write

$$\rho_{i+s_j} = \rho(x_{i+s_j}) = \rho(x_i) + \frac{\partial \rho}{\partial x} \Delta x s_j + \frac{\partial^2 \rho}{\partial x^2} \frac{\Delta x^2 s_j^2}{2} + \dots \quad (13)$$

Thus, assume we state the problem of finding the PDE approximation of (8) in form

$$\sum_{j=1}^N a_j \rho_{i+s_j} \approx \sum_{k=0}^d c_k \frac{\Delta x^k}{k!} \frac{\partial^k \rho}{\partial x^k}, \quad (14)$$

where  $d$  is the highest order of derivative (*order of continuation*) we want to use. Note that zero is also included in

the right summation, since the function itself can be used in the resulting PDE. Then the vector of unknown coefficients  $c$  can be found by inserting (13) into (14):

$$c_k = \sum_{j=1}^N a_j s_j^k \quad \forall k \in \{0, \dots, d\}. \quad (15)$$

Once (15) is solved, we write the PDE approximation to (8):

$$\frac{\partial \rho}{\partial t} = \sum_{k=0}^d c_k \frac{\Delta x^k}{k!} \frac{\partial^k \rho}{\partial x^k}. \quad (16)$$

**Remark 1** (Multidimensional systems). The method can be easily generalized to include more classes of systems, such as systems having several spatial dimensions. Multidimensionality can be accounted for by assuming that a position of a node  $\rho_i$  is described by  $x_i \in \mathbb{R}^n$  and taking multidimensional Taylor expansion at (13).

**Remark 2** (Inhomogeneous systems). Space-dependent systems can also be continualized using the same method. Indeed, choosing a unique  $d$  for all agents, we can perform continuation at positions  $x_i$  of all agents, obtaining possibly different coefficients  $c_k(x_i)$  for  $k \in \{0, \dots, d\}$ . We can then interpret them as sampled values of some continuous functions  $c_k(x)$  and approximate it by any method for continuous approximation, for example interpolation or least squares approximation.

### C. Accuracy of continuation

Procedures of discretization and continuation look very similar from the algebraic point of view, however they are qualitatively different in the way how the problem is formulated and how we should interpret their results. Indeed, the discretization step  $\Delta x$  is usually an adjustable parameter which can be set by a system engineer *arbitrarily small* to satisfy the desired performance. Thus the notion of accuracy of a discretization is used to describe how fast the solution of



the discretized equation tends to the solution of the original equation when  $\Delta x$  tends to zero.

Instead, when the original system is given by the ODE, the nodes have fixed locations, thus  $\Delta x$  is a *true constant* representing properties of an underlying physical system and it cannot be changed by an engineer. However, the higher order of continuation is taken, the better the original ODE operator (8) is approximated by the PDE (16). This intuition is supported by the following theorem, proven in [15]:

**Theorem 1.** *The spectrum of the PDE (16) converges to the spectrum of the original ODE (8) pointwise as  $d \rightarrow \infty$ .*

The convergence of spectrums is not uniform, therefore for any  $d$  at high enough frequencies the approximation will be not accurate. This effect can lead to a loss of stability properties of the original system. In particular, one of corollaries in [15] shows that if the order of continuation is a multiple of four,  $d = 4m$ , and in (16) the last gain  $c_d > 0$ , the PDE (16) becomes unstable even if the original system (8) was stable, thus an artificial instability is introduced. In the same way if  $d = 4m + 2$  and if  $c_d < 0$ , (16) is also unstable. Finally, another corollary in [15] shows that odd terms in (16) don't change stability properties since they act only on imaginary plane.

Taking high orders of continuation in (16) gives a high-order PDE which is difficult to work with. Thus it makes sense to take the smallest order which provides preservation of stability properties of (8). Having in mind the discussion above, in this paper we will use  $d = 2$ , restricting our analysis only to the systems for which  $c_2 > 0$  such that instability is not artificially introduced.

#### IV. BOUNDARY CONDITIONS

We now consider the problem of defining appropriate boundary conditions for the PDE approximation (16). Let us look at the Heat PDE:

$$\frac{\partial \rho}{\partial t} = \frac{\partial^2 \rho}{\partial x^2}. \quad (17)$$

Imagine that this equation is defined on an interval  $x \in [0, +\infty)$ , that is there is a boundary in the point  $x = 0$ .

There are two types of boundary conditions (or BC) which can be supplied to provide a well-posed boundary value problem. For example for some  $a \in \mathbb{R}$ ,

- 1) *Dirichlet* BC:  $\rho(0) = a$ ,
- 2) *Neumann* BC:  $\partial \rho / \partial x(0) = a$ .

There can also exist a linear combination of these boundary conditions, called *Robin* BC.

If the Heat Equation (17) is discretized in stencil points  $\{i-1, i, i+1\}$ , the result is

$$\dot{\rho}_i = \frac{1}{\Delta x^2} (\rho_{i-1} - 2\rho_i + \rho_{i+1}). \quad (19)$$

Assume now that there exists  $i_0 = 1$  such that  $x_{i_0-1} = 0$ . Depending on the type of boundary conditions, the equation

for the state  $\rho_1$  can be obtained by the discretization of a boundary value problem (17)-(18) in two ways:

- 1) *Dirichlet* BC:  $\dot{\rho}_1 = (a - 2\rho_1 + \rho_2) / \Delta x^2$ ,
- 2) *Neumann* BC:  $\dot{\rho}_1 = (\rho_2 - \rho_1) / \Delta x^2 - a / \Delta x$ .

Now imagine the system (17) is obtained by the continuation process from the system (19). We can notice that states of (19) are governed by the same dynamics except for the boundary state  $\rho_1$ . The question is how to recover the boundary conditions (18) for the PDE from the dynamics of  $\rho_1$  in (20).

This indeed can be done if one assumes that there exists a "ghost cell"  $\rho_0$  such that it has no dynamics, but is algebraically connected with adjacent states. With a proper definition of  $\rho_0$  the equation for  $\dot{\rho}_1$  can be represented in the same way as for other states (19) and thus has the same continuation (17). For example, algebraic equations for  $\rho_0$  representing (19)-(20) are

- 1) *Dirichlet* BC:  $\rho_0 = a$ ,
- 2) *Neumann* BC:  $\rho_0 = \rho_1 - a\Delta x$ .

The ghost cell  $\rho_0 = a$  for the case of Dirichlet BC is depicted in Fig. 5. Notice that equations (21) can be directly continualized, obtaining (18).

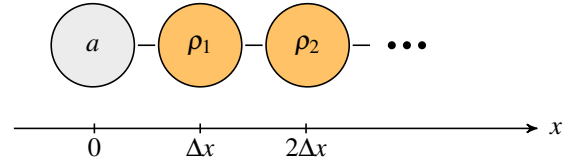


Fig. 5. Boundary of the system (19) with Dirichlet boundary condition (20), represented by a ghost cell  $\rho_0 = a$ .

This procedure can be generalized to any ODE system: once the states near boundaries change their dynamics with respect to the general governing equation, this change can be represented by "ghost cells" with algebraic dependences on the "real" states. Continualizing these algebraic equations leads to the boundary conditions for the obtained PDE.

#### V. GENERAL LINEAR NETWORK

In this section we move on to the analysis of a network system. Assume the system is given by a linear model, with  $\rho_i \in \mathbb{R}$  being the state of  $i$ -th agent, and  $x_i \in \mathbb{R}^n$  being its spatial position. We can assume that every agent  $i$  is influenced by its neighbourhood  $\mathcal{N}_i$  and also has its own dynamics:

$$\dot{\rho}_i = a_{ii}\rho_i + \sum_{j \in \mathcal{N}_i} a_{ij}\rho_j. \quad (22)$$

Note that (22) is a space-dependent multidimensional generalization of (8). Also we assume that ghost cells are added to the system (22) to ensure boundary conditions as in Section IV. A particular choice would be to have a set of boundary nodes, placed on the boundaries of the domain, with either fixed or controlled states.

Based on Theorem 1 and on the discussion in Section III-C, we choose the order of continuation  $d = 2$  to study transportation and diffusion properties of large-scale network.

Therefore, the continuation of the state  $\rho_j$  at the point  $x_i$  can be performed in the following way:

$$\begin{aligned} \rho_j = \rho(x_j) &\approx \rho(x_i) + (x_j - x_i)^T \cdot \nabla \rho + \\ &+ \frac{1}{2} (x_j - x_i)^T \frac{\partial^2 \rho}{\partial x^2} (x_j - x_i), \end{aligned} \quad (23)$$

or using the property of trace:

$$\begin{aligned} \rho_j = \rho(x_j) &\approx \rho(x_i) + (x_j - x_i)^T \cdot \nabla \rho + \\ &+ \frac{1}{2} \text{Tr} \left( (x_j - x_i)(x_j - x_i)^T \frac{\partial^2 \rho}{\partial x^2} \right), \end{aligned} \quad (24)$$

which leads to the PDE, which can be written at agents' positions as

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \left[ a_{ii} + \sum_{j \in \mathcal{N}_i} a_{ij} \right] \rho + \left[ \sum_{j \in \mathcal{N}_i} a_{ij} (x_j - x_i)^T \right] \cdot \nabla \rho + \\ &+ \text{Tr} \left( \left[ \frac{1}{2} \sum_{j \in \mathcal{N}_i} a_{ij} (x_j - x_i)(x_j - x_i)^T \right] \frac{\partial^2 \rho}{\partial x^2} \right). \end{aligned} \quad (25)$$

Define  $\lambda(x) \in \mathbb{R}$ ,  $b(x) \in \mathbb{R}^n$  and  $\varepsilon(x) \in \mathbb{R}^{n \times n}$  such that

$$\begin{aligned} \lambda(x_i) &\approx \left[ a_{ii} + \sum_{j \in \mathcal{N}_i} a_{ij} \right], \\ b(x_i) &\approx \left[ \sum_{j \in \mathcal{N}_i} a_{ij} (x_j - x_i) \right], \\ \varepsilon(x_i) &\approx \left[ \frac{1}{2} \sum_{j \in \mathcal{N}_i} a_{ij} (x_j - x_i)(x_j - x_i)^T \right], \end{aligned} \quad (26)$$

thus these functions are found by a continuous approximation of coefficients of (25). With the help of these functions we finally formulate the main continuation result:

**Theorem 2.** *The continuation of a linear network (22) is given by*

$$\frac{\partial \rho}{\partial t} = \lambda(x) \rho + b(x)^T \cdot \nabla \rho + \text{Tr} \left( \varepsilon(x) \frac{\partial^2 \rho}{\partial x^2} \right), \quad (27)$$

where  $\lambda(x)$ ,  $b(x)$  and  $\varepsilon(x)$  are given by (26).

*Remark 3.* Note that if  $a_{ij} > 0$  then the matrix inside of the trace is positive-semidefinite, which means that under suitable affine transformation of local coordinates the second-order term can be represented as a stable Laplacian diffusion. This corresponds to  $c_2 > 0$  in (16), required in Section III-C.

It is possible to derive several important corollaries for different classes of networks:

**Corollary 1** (Laplacian network). *If the original system (22) depends only on the differences of states*

$$\dot{\rho}_i = \sum_{j \in \mathcal{N}_i} a_{ij} (\rho_j - \rho_i),$$

then (27) has  $\lambda(x) \equiv 0$ .

*Proof.* This property corresponds to the fact that the network has no self-loops. For the Laplacian network  $a_{ii} = - \sum_{j \in \mathcal{N}_i} a_{ij}$ , thus by (26)  $\lambda(x) \equiv 0$ .  $\square$

**Corollary 2** (Symmetric network). *If the original system is symmetric, that is for every  $j \in \mathcal{N}_i$  there exists such  $j' \in \mathcal{N}_i$  that  $x_j - x_i = -(x_{j'} - x_i)$  and  $a_{ij} = a_{ij'}$ , then  $b(x) \equiv 0$ .*

*Proof.* Straightforward by (26).  $\square$

**Corollary 3** (Undirected regular network). *If  $a_{ij} = a_{ji}$  for all  $i, j$  and if for every  $j \in \mathcal{N}_i$  there exists such  $j' \in \mathcal{N}_i$  that  $x_j - x_i = -(x_{j'} - x_i)$ , then (27) can be represented in the form*

$$\frac{\partial \rho}{\partial t} = \lambda(x) \rho + \nabla \cdot \left( \varepsilon(x) \frac{\partial \rho}{\partial x} \right), \quad (28)$$

*Proof.* Indeed, by taking the derivative we see that

$$\nabla \cdot \left( \varepsilon(x) \frac{\partial \rho}{\partial x} \right) = (\nabla \cdot \varepsilon(x)) \cdot \nabla \rho + \text{Tr} \left( \varepsilon(x) \frac{\partial^2 \rho}{\partial x^2} \right), \quad (29)$$

and it remains to prove that  $b^T = \nabla \cdot \varepsilon$ .

Since  $a_{ij} = a_{ji}$ , we can assume that there exists some continuous function  $\alpha(x, \bar{n})$  dependent on the coordinate  $x$  and the direction  $\bar{n}$  even with respect to the direction such that

$$a_{ij} = \alpha \left( \frac{x_i + x_j}{2}, \frac{x_j - x_i}{\|x_j - x_i\|} \right) = \alpha \left( \frac{x_i + x_j}{2}, \frac{x_i - x_j}{\|x_i - x_j\|} \right) = a_{ji}$$

Denote  $\bar{n}_j = (x_i - x_j) / \|x_i - x_j\|$ . Further, define  $y = x_i$  to be the point where the function  $\alpha$  is investigated, thus  $\alpha((x_i + x_j)/2, \bar{n}_j) = \alpha(y + (x_j - x_i)/2, \bar{n}_j)$ . We can now take the Taylor expansion of this function with respect to the coordinate:

$$\alpha \left( y + \frac{x_j - x_i}{2}, \bar{n}_j \right) \approx \alpha(y, \bar{n}_j) + \frac{1}{2} \nabla \alpha(y, \bar{n}_j) \cdot (x_j - x_i).$$

Inserting this expansion into the definition of  $b(x)$  we obtain

$$\begin{aligned} b(y)^T &= \sum_{j \in \mathcal{N}_i} \alpha(y, \bar{n}_j) (x_j - x_i)^T + \\ &+ \frac{1}{2} \sum_{j \in \mathcal{N}_i} \nabla \alpha(y, \bar{n}_j) \cdot (x_j - x_i) (x_j - x_i)^T = \\ &= \frac{1}{2} \sum_{j \in \mathcal{N}_i} \nabla \alpha(y, \bar{n}_j) \cdot (x_j - x_i) (x_j - x_i)^T, \end{aligned} \quad (30)$$

since the first sum vanishes because for every  $j$  there exists  $j'$  such that  $(x_j - x_i) = -(x_{j'} - x_i)$  and  $\alpha(y, \bar{n}_j) = \alpha(y, \bar{n}_{j'})$ . Now, analyzing  $\varepsilon(x)$ , we get

$$\begin{aligned} \varepsilon(y) &= \frac{1}{2} \sum_{j \in \mathcal{N}_i} \alpha(y, \bar{n}_j) (x_j - x_i)(x_j - x_i)^T + \\ &+ \frac{1}{4} \sum_{j \in \mathcal{N}_i} (x_j - x_i) \nabla \alpha(y, \bar{n}_j) (x_j - x_i)(x_j - x_i)^T = \\ &= \frac{1}{2} \sum_{j \in \mathcal{N}_i} \alpha(y, \bar{n}_j) (x_j - x_i)(x_j - x_i)^T, \end{aligned} \quad (31)$$

where the second sum vanishes by the same reasons. Now it is clear that taking the divergence of (31) with respect to  $y$  one ends up with (30), which finishes the proof.  $\square$

## VI. CONCLUSION

In this paper we used a recently developed continuation method to transform a general linear spatially-distributed network in a single second-order linear PDE with space-dependent coefficients. We showed how the properties of the original network can affect the resulting PDE. Further we demonstrated the effectiveness of such approach by stabilizing a network of semiconductor lasers using a continuous PDE model and an explicit PDE-based backstepping boundary control.

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